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On the Chebyshev collocation spectral approach to stability of fluid flow in a porous medium

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SUMMARY

In this paper, the temporal development of small disturbances in a pressure-driven fluid flow through a channel filled with a saturated porous medium is investigated. The Brinkman flow model is employed in order to obtain the basic flow velocity distribution. Under normal mode assumption, the linearized governing equations for disturbances yield a fourth-order eigenvalue problem, which reduces to the well-known Orr–Sommerfeld equation in some limiting cases solved numerically by a spectral collocation technique with expansions in Chebyshev polynomials. The critical Reynolds number Re_c, the critical wave number α_c, and the critical wave speed c_c are obtained for a wide range of the porous medium shape factor parameter S. It is found that a decrease in porous medium permeability has a stabilizing effect on the fluid flow. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Transport processes through porous media play important roles in diverse applications, such as in geothermal operations, petroleum industries, thermal insulation, design of solid-matrix heat exchangers, chemical catalytic reactors, and many others [1, 2]. Theoretical consideration of fluid flow in porous media has received great attention in recent years. Most of the earlier studies [3, 4] were based on Darcy's law, which states that the volume-averaged velocity is proportional to the pressure gradient. The Darcy model is shown to be valid under the conditions of low velocities

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and small porosity [5]. However, in many practical situations the porous medium is bounded by an impermeable wall, has higher flow rates, and reveals non-uniform porosity distribution in the near wall region, making Darcy's law inapplicable. To model a real physical situation better the Brinkman flow model is employed, since it can predict hydraulics through such hyperporous media as noted by Nield and Bejan [6]. The Brinkman model also takes into account the presence of a solid boundary through the addition of a viscous term in Darcy's law and, furthermore, it is generally applicable for porous media with both low and high permeabilities [7]. The effects of variable viscosity on the instability of flow and temperature fields in a water-saturated porous medium are discussed by Kassoy and Zebib [5], Straus and Schubert [4], and Gray *et al*. [8]. In all these studies, theoretical investigation on the temporal stability of fluid flow in a saturated porous medium with respect to the Brinkman model has not been conducted.

Motivated by scarcity of literature on the application of the Chebysev collocation spectral method on stability analysis of fluid flow in porous media, the temporal development of small disturbances in a channel filled with a saturated porous medium is investigated. The Brinkman model is employed in order to obtain the flow basic velocity profile and the linear stability analysis is performed. The resulting fourth-order eigenvalue problem, which reduces to a well-known Orr–Sommerfeld equation under some limiting cases, is solved numerically using the Chebyshev spectral collocation method. The use of spectral methods to investigate the stability of various fluid flow problems has increased in recent years. The popularity of spectral methods comes from the fact that they have been proven to produce more accurate results than the finite difference and finite element numerical schemes [9–12]. The paper is structured as follows. In Section 2, the problem is formulated and the solution for the steady basic flow is obtained. The eigenvalue problem for temporal development of small disturbances is derived in Section 3. In Section 4, the Chebyshev spectral collocation numerical technique is employed to solve the resulting eigenvalue problem and the pertinent results are discussed quantitatively in Section 5.

2. MATHEMATICAL FORMULATION

Consider the flow of an incompressible viscous fluid in a parallel channel filled with a saturated porous medium (see Figure 1).

In two dimensions, the governing equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{S^2 u}{Re} \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{S^2 v}{Re} \tag{3}$$

where x and y are the streamwise and normal coordinates, respectively; u and v are the streamwise and normal velocity, respectively; t is the time; P is the pressure; Re is the Reynolds number; and

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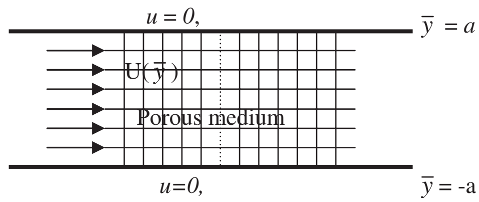

Figure 1. Geometry of the problem.

S is the shape factor. The flow quantities in Equations (1)–(3) have been non-dimensionalized as follows:

$$u=\frac{\bar{u}}{U_0},\quad v=\frac{\bar{v}}{U_0},\quad t=\frac{U_0\bar{t}}{a},\quad x=\frac{\bar{x}}{a},\quad y=\frac{\bar{y}}{a},\quad P=\frac{\bar{P}}{\rho U_0^2}$$
$$S^2=\frac{1}{Da},\quad Da=\frac{k}{a^2},\quad Re=\frac{U_0 a}{\upsilon} \tag{4}$$

where a is the channel characteristic half-width, k the permeability parameter, U_0 the characteristic fluid velocity, ρ the fluid density, υ the kinematic fluid viscosity, and Da the Darcy number. The basic steady state of the flow system corresponds to a parallel flow with velocities $u=U(y)$ and $v=0$. The equation and the boundary conditions describing the basic state are [7, 13, 14]

$$\frac{\mathrm{d}^2U}{\mathrm{d}y^2}-S^2U=-A,\quad \frac{\mathrm{d}U}{\mathrm{d}y}(0)=0,\quad U(1)=0 \tag{5}$$

The solution is given by

$$U(y,S\gg 1)=\frac{A}{S^2}\left(1-\frac{\cosh(Sy)}{\cosh(S)}\right)\quad \text{(small Darcy number)} \tag{6}$$

$$U(y,S\ll 1)\to -\frac{A}{2}(y^2-1)-\frac{AS^2}{24}(y^2-1)(y^2-5)+O(S^4)\quad \text{(large Darcy number)} \tag{7}$$

where $A=-Re\partial P/\partial x$. The zeroth-order solution in Equation (7) is the familiar one that corresponds to the plane Poiseuille flow, i.e. for the very large Darcy number case and in the limit of $\mathrm{S}\to 0, U(y)\to -A(y^2-1)/2$.

3. STABILITY ANALYSIS

In the stability analysis, two-dimensional disturbances will be considered, which implies that Squire's transformation [11, 15] is applicable. Introducing small disturbances to the basic flow as follows:

$$u(x,y,t)=U(y)+\hat{u}(x,y,t),\quad v(x,y,t)=\hat{v}(x,y,t),\quad p(x,y,t)=P(x)+\hat{p}(x,y,t) \tag{8}$$

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where $\hat{u}$, $\hat{v}$, and $\hat{p}$ are very small. Equation (8) is then substituted into Equations (1)–(3) and the nonlinear terms are neglected. We obtain

$$\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=0 \tag{9}$$

$$\frac{\partial \hat{u}}{\partial t}+U\frac{\partial \hat{u}}{\partial x}+\hat{v}\frac{\mathrm{d}U}{\mathrm{d}y}=-\frac{\partial \hat{p}}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 \hat{u}}{\partial x^2}+\frac{\partial^2 \hat{u}}{\partial y^2}\right)-\frac{S^2\hat{u}}{Re} \tag{10}$$

$$\frac{\partial \hat{v}}{\partial t}+U\frac{\partial \hat{v}}{\partial x}=-\frac{\partial \hat{p}}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 \hat{v}}{\partial x^2}+\frac{\partial^2 \hat{v}}{\partial y^2}\right)-\frac{S^2\hat{v}}{Re} \tag{11}$$

Following Orszag [10], we seek a normal mode solution for Equations (9)–(11) defined in terms of a stream function as

$$\psi(x,y,t)=\phi(y)\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{12}$$

where $\phi(y)$ is the amplitude function and c,α are the disturbances wave speed and wave number, respectively. The disturbance velocity components can be expressed as follows:

$$\hat{u}=\frac{\partial \psi}{\partial y}=\phi'(y)\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{13}$$

$$\hat{v}=-\frac{\partial \psi}{\partial x}=-\mathrm{i}\alpha\phi(y)\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{14}$$

where the prime symbol denotes differentiation with respect to y. Substituting Equations (13)–(14) into Equations (9)–(11) and eliminating the pressure terms yield

$$(U-C)(\phi''-\alpha^2\phi)-U''\phi=\frac{1}{\mathrm{i}\alpha Re}(\phi^{\mathrm{iv}}-(S^2+2\alpha^2)\phi''+(\alpha^4+S^2\alpha^2)) \tag{15}$$

with the boundary conditions

$$\begin{aligned}\phi(-1)&=\phi'(-1)=0\\ \phi(1)&=\phi'(1)=0\end{aligned} \tag{16}$$

It is noteworthy that Equation (15) reduces to the classical Orr–Sommerfeld equation [15] when $S=0$, which corresponds to a plane-Poiseuille flow situation. In order to find a non-trivial function ϕ satisfying Equation (15) with boundary conditions (16), the parameters α, Re, S, and c must satisfy a certain complex eigenvalue relation, say

$$F(\alpha,c,S,Re)=0 \tag{17}$$

For temporal development of the disturbances, α is real and c is complex, which can be expressed as

$$c=c_r(\alpha,S,Re)+\mathrm{i}c_i(\alpha,S,Re) \tag{18}$$

The imaginary part of Equation (18) determines whether the disturbances grow or decay. When $\alpha c_i>0$ the disturbances grow; when $\alpha c_i=0$ they neither grow nor decay, and in this case the disturbance modes are said to be neutrally stable.

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4. COMPUTATIONAL APPROACH

The eigenvalue problem in Equations (15)–(16) is solved using the Chebyshev spectral collocation method where the solution of the differential equation and its boundary conditions are expanded as a finite series in the Chebyshev polynomials of the form

$$\phi(y)\approx\phi_N(y_j)=\sum_{k=0}^{N}\tilde{\phi}\mathrm{T}_k(y_j),\quad j=0,1,\ldots,N \tag{19}$$

where T_k is the kth-Chebyshev polynomial defined by

$$T_0(y)=1,\quad T_1(y)=y,\quad T_{k+1}(y)-2yT_k(y)+T_{k-1}(y)=0\quad(-1\leqslant y\leqslant 1) \tag{20}$$

$\tilde{\phi}_k$ represents the unknown coefficients and are the Gauss–Lobatto collocation points [16] on the interval $[-1,1]$ defined by

$$y_j=\cos\frac{\pi j}{N},\quad j=0,1,\ldots,N \tag{21}$$

Substituting Equation (21) into Equation (19) and requiring that the differential equation (15) be satisfied at the N collocation points, we obtain $(N+1)x(N+1)$ algebraic equations that form the eigenvalue problem:

$$E\phi=cB\phi \tag{22}$$

where

$$\phi^{\mathrm{T}}=(\tilde{\phi}_0,\tilde{\phi}_1,\ldots,\tilde{\phi}_N) \tag{23}$$

is the transpose of the column vector ϕ. The clamped boundary conditions are incorporated explicitly in the first two and last rows of the matrices E and B by setting

$$E(m,n)=\begin{cases}1, & m=n=0\\ 0, & m=0,n=1,\ldots,N\\ \sum_{n=0}^{N}D_{0n}, & m=1,\ \ n=0,\ldots,N\\ \tilde{E}(m,n), & m=1,\ldots,N-2,\ \ n=0,\ldots,N\\ \sum_{n=0}^{N}D_{Nn}, & m=N-1,\ \ n=0,\ldots,N\\ 0, & m=N,\ \ n=1,\ldots,N-1\\ 1, & m=N,\ \ n=N\end{cases} \tag{24}$$

$$B(m,n)=\begin{cases}0, & m=0,1,N-1,N,\ \ n=0,\ldots,N\\ \tilde{B}(m,n), & m=2,\ldots,N-2,\ \ n=0,\ldots,N\end{cases} \tag{25}$$

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Table I. Computation showing the eigenvalue of the most unstable mode ($Re=20\,000$, $\alpha=1.0$, $A=2$).

S	Wavespeed (c)
0.000000	0.20966327758363+0.00330625189812**I**
0.100000	0.20894714656099+0.00329946464059**I**
0.200000	0.20683099825483+0.00327550439674**I**
0.300000	0.20340796033453+0.00322443525286**I**
0.400000	0.19882195076488+0.00313228046581**I**
0.500000	0.19325306790742+0.00298405088132**I**
0.600000	0.18690107771857+0.00276666982581**I**
0.700000	0.17996955903920+0.00247113278055**I**
0.800000	0.17265265209722+0.00209359713090**I**
0.900000	0.16512546638887+0.00163543541538**I**
1.000000	0.15753834592603+0.00110252125421**I**

Table II. Computations of the critical values at which unstable modes begin to exist ($A=2$).

S	α_c	Re_c	c_c
0.0	1.02052	5772.2283	0.263997+0.000000**I**
0.1	1.01986	5832.2973	0.262559+0.000000**I**
0.2	1.01781	6015.0334	0.258313+0.000000**I**
0.3	1.01492	6328.7057	0.251535+0.000000**I**
0.4	1.01074	6787.3070	0.242499+0.000000**I**
0.5	1.00561	7411.1295	0.231650+0.000000**I**
0.6	0.99966	8227.4284	0.219445+0.000000**I**
0.7	0.99307	9271.2789	0.206344+0.000000**I**
0.8	0.98608	10586.3898	0.192782+0.000000**I**

where

$$\tilde{E}=\frac{\mathrm{i}}{\alpha Re}(D^4-[S^2+2\alpha^2]D^2+(\alpha^4+S^2\alpha^2)I)+U(D^2-\alpha^2 I)-U'' \tag{26}$$

$$\tilde{B}=(D^2-\alpha^2 I) \tag{27}$$

$U=\text{diag}\,[U(y_j)]$, I is the $(N+1)x(N+1)$ identity matrix, and D is the usual differential matrix (cf. Canuto *et al*. [17]). Here diag[ ] means that the entries are placed on the main diagonal of an $(N+1)x(N+1)$ matrix with the rest of the entries being zero, which usually results in matrix B becoming singular. The problem is avoided by employing the idea of Weidemann and Reddy [18], i.e. using Hermite interpolating polynomials that satisfy the boundary conditions; thus, we obtain

$$\tilde{\phi}_0=0, \quad \sum_{n=0}^{N} D_{0n}\tilde{\phi}_n=0 \quad \text{on } y=1 \tag{28}$$

$$\tilde{\phi}_N=0, \quad \sum_{n=0}^{N} D_{Nn}\tilde{\phi}_n=0 \quad \text{on } y=-1 \tag{29}$$

5. NUMERICAL RESULTS AND DISCUSSION

Here, we emphasis that the porous medium permeability decreases with increasing positive values of the shape factor parameter (S). The eigensolutions of the generalized eigenvalue problem (25)–(28) obtained numerically are presented in this section. The numerical solutions have been verified for correctness by comparing with the results obtained by Orszag [10] for $S=0$ and perfect agreement is observed.

Table I shows the numerical results for the eigenvalues of the most unstable mode for increasing values of S at fixed values of A, α, and Re. It is interesting to note that a slight increase in the values of S has the effect of decreasing the real and imaginary parts of the wavespeed. This shows that a decrease in the porous medium permeability has stabilizing effects on the flow. Table II

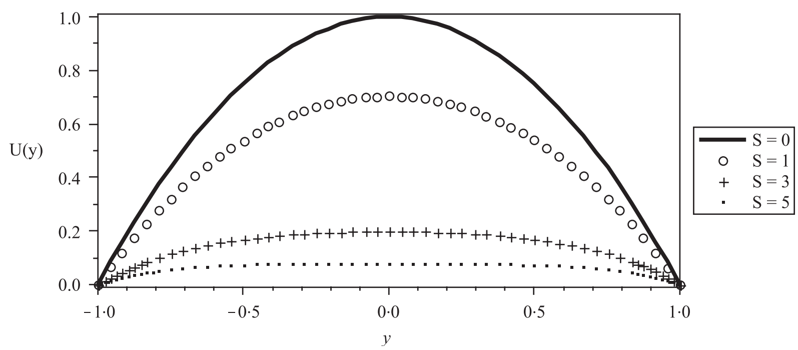

Figure 2. Basic velocity profile with increasing values of S.

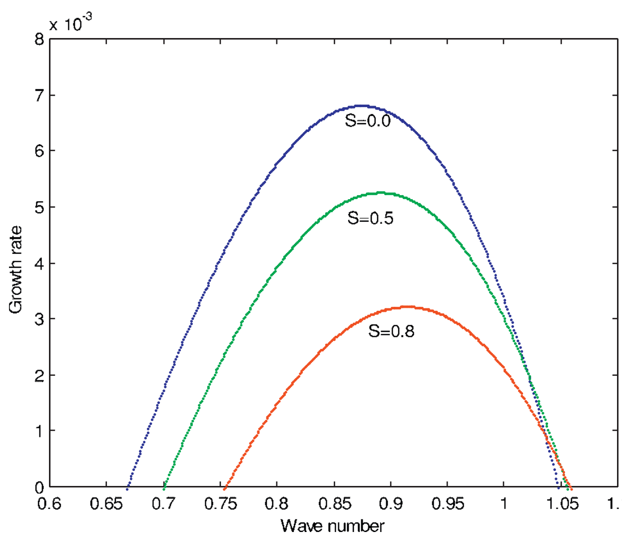

Figure 3. Growth rate (αc_i) against wavenumber (α).

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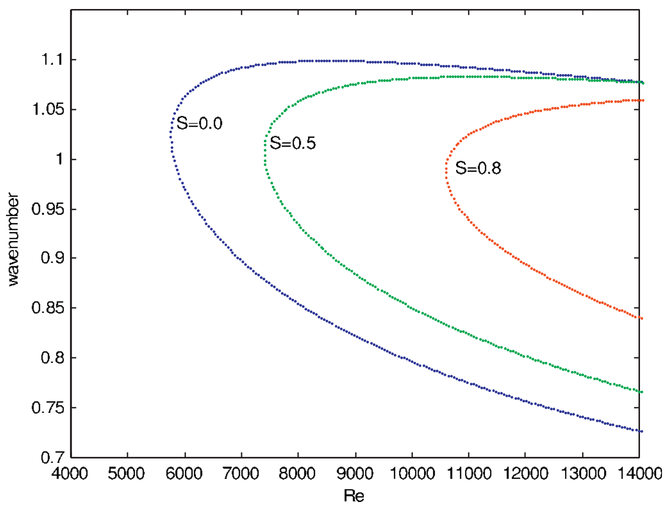

Figure 4. Marginal stability curves for $A=2$, $S=0.0$, 0.5, and 0.8.

shows the critical values of the wavenumber α_c, wavespeed c_c, and Reynolds numbers Re_c at which unstable modes begin to exist for varying values of S. We observe that an increase in S leads to an increase in the critical Reynolds number and a slight decrease in the critical wavespeed. This means that the stable region in (Re, α) plane increases as the shape factor S increases (see Figure 4). Figure 2 illustrates the axial velocity profiles; a parabolic plane-Poiseuille profile is observed for $S=0$ with maximum value along the centerline and minimum at the wall. However, for increasing values of $S>0$, the fluid velocity decreases and flattens out due to a gradual decrease in the porous medium permeability. Figure 3 shows the variation in the growth rate of the most unstable mode against the wavenumber. We observe that increasing values of S have the effect of damping the disturbances and therefore eliminating the growth of any small disturbances in the flow field.

6. CONCLUSION

The Chebyshev spectral collocation method is employed to investigate the temporal development of small disturbances in a channel filled with a saturated porous medium. We obtained accurately the critical Reynolds number Re_c, critical wave number α_c, and the critical wave speed c_c for a wide range of porous medium shape factor parameter S. It is observed that increasing values of shape factor S that indicates a decrease in porous medium permeability have a stabilizing effect on the fluid flow.

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